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EXPLICIT cocycle formulas on finite abelian groups with applications to braided linear Gr-categories and Dijkgraaf–Witten invariants

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We provide explicit and unified formulas for the cocycles of all degrees on the normalized bar resolutions of finite abelian groups. This is achieved by constructing a chain map from the normalized bar resolution to a Koszul-like resolution for any given finite abelian group. With a help of the obtained cocycle formulas, we determine all the braided linear Gr-categories and compute the Dijkgraaf–Witten Invariants of the n-torus for all n.

Keywords: group cocycle; Gr-category; DW invariant

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1. Introduction

Throughout, let $\Bbbk$ be an algebraically closed field of characteristic zero and let $\Bbbk^*$ denote the multiplicative group $\Bbbk - \{0\}$. Unless otherwise specified, all algebraic structures and linear operations are over $\Bbbk$. Our main aim is to provide explicit and unified formulas for the cocycles on the normalized bar resolutions (normalized cocycles) of finite abelian groups. Some applications to braided linear Gr-categories and Dijkgraaf–Witten Invariants (DW invariant) are also considered.

The cohomology groups of finite abelian groups are computable thanks to the well known Lyndon-Hochschild-Serre spectral sequence [15, 23]. However, the explicit formulas of normalized cocycles are not clear in literatures. Such explicit formulas of normalized cocycles, instead of the cohomology groups, are necessary in many respects of mathematics and physics. Besides the connections to braided linear Gr-categories and DW invariants involved in the present paper, normalized 2-cocycles are necessary in projective representation theory of finite groups [11, 22]; normalized 3-cocycles are indispensable in the classification program of pointed finite tensor categories and quasi-quantum groups [8, 9, 12, 17, 19, 20]; normalized cocycles of all degrees are very important in the theory of symmetry protected topological orders [2, 3, 29].

Our approach of formulating the normalized cocycles is straightforward and elementary. First we construct a Koszul-like resolution of a finite abelian group G by tensoring the minimal resolutions of cyclic factors of G and give a complete set of representatives of cocycles for this resolution. Then we construct a chain map from the normalized bar resolution to this Koszul-like resolution. Finally we get the desired explicit and unified formulas of normalized cocycles on G by pulling back those on the Koszul-like resolution along the chain map. We remark that, in principle, the method of Lyndon-Hochschild-Serre spectral sequence may also help one formulate explicit forms of normalized cocycles with nearly as much effort as we need here.

Here is a brief description of the content. In § 2, we provide formulas of normalized cocycles of all degrees on any finite abelian groups. In § 3, we use the formula of normalized 3-cocycles to determine the braided monoidal structures on linear Gr-categories. In § 4, we give a formula for the DW invariant of the n-torus for all n and obtain the dimension formula for irreducible projective representations of an arbitrary finite abelian group.

2. Explicit formulas of normalized cocycles on finite abelian groups

In this section, we use freely the concepts and notations about group cohomology in the book [30] of Weibel. Let G be a group and $(B_\bullet, \partial_\bullet)$ be its normalized bar resolution. Applying $\mathrm{Hom}_{\mathbb{Z}G}(-, \Bbbk^*)$ one gets a complex $(B_\bullet^*, \partial_\bullet^*)$. Denote the group of normalized n-cocycles by $Z^n(G, \Bbbk^*)$, which is $\mathrm{Ker}\, \partial_n^*$. In general, it is hard to determine $Z^n(G, \Bbbk^*)$ directly as the normalized bar resolution is far too large. Our strategy of overcoming this is to get first a simpler resolution of G whose cocycles are easy to compute and then construct a chain map from the normalized bar resolution to it which will help to determine $Z^n(G, \Bbbk^*)$ eventually.

2.1. A Koszul-like resolution

From now on let G be a finite abelian group. Write $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ where $m_i | m_{i+1}$ for $1 \leqslant i \leqslant n-1$ and for every $\mathbb{Z}_{m_i}$ fix a generator g_i for $1 \leqslant i \leqslant n$. It is well known that the following periodic sequence is a free resolution of the trivial $\mathbb{Z}_{m_i}$-module $\mathbb{Z}$:

$$\cdots \longrightarrow \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{T_i} \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{N_i} \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{T_i} \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{N_i} \mathbb{Z} \longrightarrow 0, \qquad (2.1)$$

where $T_i = g_i - 1$ and $N_i = \sum_{j=0}^{m_i-1} g_i^j$.

Consider the tensor product of the above periodic resolutions of the cyclic factors of G. The resulting complex, denoted by $(K_\bullet, d_\bullet)$, is as follows. For each sequence $a_1, \ldots, a_n$ of nonnegative integers, let $\Phi(a_1, \ldots, a_n)$ be a free generator in degree $a_1 + \cdots + a_n$. Thus

$$K_m := \bigoplus_{a_1+\cdots+a_n=m} (\mathbb{Z}G)\Phi(a_1, \ldots, a_n).$$

For all $1 \leqslant i \leqslant n$, define

$$d_i(\Phi(a_1,\dots,a_n)) = \begin{cases} 0, & a_i = 0; \\ (-1)^{\sum_{l<i} a_l} N_i \Phi(a_1,\dots,a_i-1,\dots,a_n), & 0 \neq a_i \text{ even}; \\ (-1)^{\sum_{l<i} a_l} T_i \Phi(a_1,\dots,a_i-1,\dots,a_n), & 0 \neq a_i \text{ odd}. \end{cases}$$

The differential d is set to be $d_1 + \cdots + d_n$. Then $(K_\bullet, d_\bullet)$ is a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. The main goal of this subsection is to determine the explicit cocycles of this Koszul-like resolution.

For the convenience of the exposition, we fix some notations before moving on. For any $1 \leqslant r_1 < \cdots < r_l \leqslant n$, define $\Phi_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} := \Phi(0,\dots,\lambda_1,\dots,\lambda_l,\dots,0)$ where $\lambda_i \geqslant 1$ lies in the r_i-th position. If $\lambda_i = 1$ for some $1 \leqslant i \leqslant l$, sometimes we drop it for brevity. It is clear that any cochain $f \in \mathrm{Hom}_{\mathbb{Z}G}(K_k, \Bbbk^*)$ is uniquely determined by its values on $\Phi_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}$. Write $f_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} = f(\Phi_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}})$.

Theorem 2.1. *The following*

$$\left\{ f \in \mathrm{Hom}_{\mathbb{Z}G}(K_k, \Bbbk^*) \;\middle|\; \begin{array}{l} f_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} = 1 \text{ if } \lambda_1 \text{ is even,} \\ f_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} = \zeta_{m_{r_1}}^{a_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}} \text{ if } \lambda_1 \text{ is odd} \\ \text{and } 0 \leqslant a_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} < m_{r_1} \text{ for } 1 \leqslant r_1 < \cdots < r_l \leqslant n \\ \text{where } \lambda_1 + \cdots + \lambda_l = k,\ \lambda_i \geqslant 1 \text{ for } 1 \leqslant i \leqslant l \end{array} \right\} \tag{2.2}$$

makes a complete set of representatives of k-cocycles of the complex $(K_\bullet^*, d_\bullet^*)$.

Proof. Suppose $f \in \mathrm{Hom}_{\mathbb{Z}G}(K_k, \Bbbk^*)$ is a k-cocycle. We will show that f is cohomologous to one in (2.2). Let $g \in \mathrm{Hom}_{\mathbb{Z}G}(K_{k-1}, \Bbbk^*)$ be a $(k-1)$-cochain given by $g_{r_1^{\mu_1}\cdots r_l^{\mu_l}} = 1$ if μ_1 is even and $g_{r_1^{\mu_1}\cdots r_l^{\mu_l}} = (f_{r_1^{\mu_1+1}\cdots r_l^{\mu_l}})^{1/m_{r_1}}$ if μ_1 is odd. Consider $f' = f - d^*g$. Then clearly $f'_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} = 1$ if λ_1 is even. If λ_1 is odd, then by the cocycle condition for f' we have

$$(f'_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}})^{m_{r_1}} \prod_{\substack{2 \leqslant i \leqslant l \\ \lambda_i \text{ even}}} (f'_{r_1^{\lambda_1+1}\cdots r_i^{\lambda_i-1}\cdots r_l^{\lambda_l}})^{(-1)^{\sum_{j<i}\lambda_j+1} m_{r_i}} = 1.$$

Hence $(f'_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}})^{m_{r_1}} = 1$, so $f'_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} = \zeta_{m_{r_1}}^{a_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}}$ for some $0 \leqslant a_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} < m_{r_1}$.

Suppose that f_1 and f_2 are two cocycles in (2.2) satisfying $f_1 - f_2 = d^*h$ for some $(k-1)$-cochain $h \in \mathrm{Hom}_{\mathbb{Z}G}(K_{k-1}, \Bbbk^*)$. Similarly as above, after subtracting a $(k-1)$-coboundary from h, we can assume that $h_{r_1^{\mu_1}\cdots r_l^{\mu_l}} = 1$ if μ_1 is even. If λ_1 is even, then

$$\begin{aligned} (f_1' - f_2')_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} &= (h_{r_1^{\lambda_1-1}\cdots r_l^{\lambda_l}})^{m_{r_1}} \prod_{\substack{2 \leqslant i \leqslant l \\ \lambda_i \text{ even}}} (h_{r_1^{\lambda_1}\cdots r_i^{\lambda_i-1}\cdots r_l^{\lambda_l}})^{(-1)^{\sum_{j<i}\lambda_j} m_{r_i}} \\ &= (h_{r_1^{\lambda_1-1}\cdots r_l^{\lambda_l}})^{m_{r_1}} = 1. \end{aligned}$$

If λ_1 is odd, then by the preceding equation and the condition $m_i|m_{i+1}$ for $1 \leqslant i \leqslant n-1$ we have

$$(f_1 - f_2)_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} = \prod_{\substack{2 \leqslant i \leqslant l \\ \lambda_i \text{ even}}} (h_{r_1^{\lambda_1}\cdots r_i^{\lambda_i - 1}\cdots r_l^{\lambda_l}})^{(-1)^{\sum_{j<i}\lambda_j} m_{r_i}} = 1.$$

Hence $f_1 = f_2$. □

Corollary 2.2. *If $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ where $m_i|m_{i+1}$ for $1 \leqslant i \leqslant n-1$, then*

$$\mathrm{H}^k(G, \Bbbk^*) = \prod_{r=1}^{n} \mathbb{Z}_{m_r}^{\sum_{j=1}^{k}(-1)^{k+j}\binom{n-r+j-1}{j-1}}.$$

Proof. By theorem 2.1, $\mathrm{H}^k(G, \Bbbk^*) = \prod_{r=1}^{n} \mathbb{Z}_{m_r}^{N_{k,r}}$ where

$$\begin{aligned} N_{k,r} &= \#\{(r_2, \ldots, r_l, \lambda_1, \ldots, \lambda_l) \in \mathbb{N}^{2l-1} | r < r_2 < \cdots < r_l \\ &\qquad \leqslant n, \lambda_1 + \cdots + \lambda_l = k, \lambda_1 \text{ odd}\} \\ &= \sum_{l=1}^{k} \binom{n-r}{l-1} \#\{(\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l | \lambda_1 + \cdots + \lambda_l = k, \lambda_1 \text{ odd}\}. \end{aligned}$$

Denote $s_{k,l} = \#\{(\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l | \lambda_1 + \cdots + \lambda_l = k, \lambda_1 \text{ odd}\}$ and $t_{k,l} = \#\{(\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l | \lambda_1 + \cdots + \lambda_l = k, \lambda_1 \text{ even}\}$. Then $s_{k,l} = t_{k+1,l}$ and $s_{k,l} + t_{k,l} = \binom{k-1}{l-1}$. Hence

$$N_{k,r} + N_{k-1,r} = \sum_{l=1}^{k} \binom{n-r}{l-1}\binom{k-1}{l-1} = \binom{n-r+k-1}{k-1}.$$

Therefore, $N_{k,r} = \sum_{j=1}^{k}(-1)^{k+j}\binom{n-r+j-1}{j-1}$. □

2.2. A chain map from $(B_\bullet, \partial_\bullet)$ to $(K_\bullet, d_\bullet)$

We need some more notations to present our chain map. For any positive integers s and r, let $[s/r]$ denote the integer part of s/r and let s'_r denote the remainder of the division of s by r. When there is no risk of confusion, we omit the subscript in s'_r. It is easy to observe that

$$\left[\frac{s + t'_r}{r}\right] = \left[\frac{s + t - [t/r]r}{r}\right] = \left[\frac{s+t}{r}\right] - \left[\frac{t}{r}\right] \tag{2.3}$$

for any three natural numbers s, t and r. We need the following technical lemma for later discussions.

LEMMA 2.3. *Let r be a positive integer. For any $2l+1$ natural numbers $a_1,\ a_2,\ldots,a_{2l+1}$, we have the following equation*

$$\begin{aligned}
&\sum_{i=1}^{l}\left[\frac{a_{2l+1}+a_{2l}}{r}\right]\cdots\left[\frac{a_{2i+3}+a_{2i+2}}{r}\right]\\
&\quad\times\left(\left[\frac{a_{2i+1}+(a_{2i}+a_{2i-1})'}{r}\right]-\left[\frac{(a_{2i+1}+a_{2i})'+a_{2i-1}}{r}\right]\right)\\
&\quad\cdot\left[\frac{a_{2i-2}+a_{2i-3}}{r}\right]\cdots\left[\frac{a_2+a_1}{r}\right]\\
&\quad=\left[\frac{a_{2l+1}+a_{2l}}{r}\right]\cdots\left[\frac{a_3+a_2}{r}\right]-\left[\frac{a_{2l}+a_{2l-1}}{r}\right]\cdots\left[\frac{a_2+a_1}{r}\right].
\end{aligned}$$

Proof. By (2.3), we have

$$\begin{aligned}
&\left[\frac{a_{2i+1}+(a_{2i}+a_{2i-1})'}{r}\right]-\left[\frac{(a_{2i+1}+a_{2i})'+a_{2i-1}}{r}\right]\\
&\quad=\left[\frac{a_{2i+1}+a_{2i}}{r}\right]-\left[\frac{a_{2i}+a_{2i-1}}{r}\right].
\end{aligned}$$

Then the lemma follows by an obvious elimination of consecutive terms. □

Now we are ready to give a chain map from the normalized bar resolution $(B_\bullet,\partial_\bullet)$ to the Koszul-like resolution $(K_\bullet,d_\bullet)$. Recall that B_m is the free $\mathbb{Z}G$-module on the set of all symbols $[h_1,\ldots,h_m]$ with $h_i\in G$ and $m\geqslant 1$. In the case $m=0$, the symbol $[\,]$ denote $1\in\mathbb{Z}G$ and the map $\partial_0=\epsilon:\ B_0\to\mathbb{Z}$ sends $[\,]$ to 1. For a more concise formulation, denote $(g_i)_r:=\sum_{j=0}^{r-1}g_i^j$ for $1\leqslant i\leqslant n$ in the following.

The first four terms of the chain map, which will be used for later applications, are as follows:

$$\begin{aligned}
&F_1: B_1\longrightarrow K_1\\
&\qquad [g_1^{i_1}\cdots g_n^{i_n}]\mapsto\sum_{s=1}^{n}g_1^{i_1}\cdots g_{s-1}^{i_{s-1}}(g_s)_{i_s}\Phi_s;\\
&F_2: B_2\longrightarrow K_2\\
&\qquad [g_1^{i_1}\cdots g_n^{i_n},g_1^{j_1}\cdots g_n^{j_n}]\mapsto\sum_{s=1}^{n}g_1^{i_1+j_1}\cdots g_{s-1}^{i_{s-1}+j_{s-1}}\left[\frac{i_s+j_s}{m_s}\right]\Phi_{s^2}\\
&\qquad\qquad -\sum_{1\leqslant s<t\leqslant n}g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}g_1^{j_1}\cdots g_{s-1}^{j_{s-1}}(g_s)_{j_s}(g_t)_{i_t}\Phi_{st};\\
&F_3: B_3\longrightarrow K_3\\
&\qquad [g_1^{i_1}\cdots g_n^{i_n},g_1^{j_1}\cdots g_n^{j_n},g_1^{k_1}\cdots g_n^{k_n}]
\end{aligned}$$

$$
\begin{aligned}
&\mapsto \sum_{r=1}^{n} \left[\frac{j_r+k_r}{m_r}\right] g_1^{j_1+k_1}\cdots g_{r-1}^{j_{r-1}+k_{r-1}} g_1^{i_1}\cdots g_{r-1}^{i_{r-1}}(g_r)_{i_r}\Phi_{r^3}\\
&\quad + \sum_{1\leqslant r<t\leqslant n} \left[\frac{j_r+k_r}{m_r}\right] g_1^{j_1+k_1}\cdots g_{r-1}^{j_{r-1}+k_{r-1}} g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}(g_t)_{i_t}\Phi_{r^2t}\\
&\quad + \sum_{1\leqslant r<t\leqslant n} \left[\frac{i_t+j_t}{m_t}\right] g_1^{i_1+j_1}\cdots g_{t-1}^{i_{t-1}+j_{t-1}} g_1^{k_1}\cdots g_{r-1}^{k_{r-1}}(g_r)_{k_r}\Phi_{rt^2}\\
&\quad - \sum_{1\leqslant r<s<t\leqslant n} g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}(g_t)_{i_t} g_1^{j_1}\cdots g_{s-1}^{j_{s-1}}(g_s)_{j_s} g_1^{k_1}\cdots g_{r-1}^{k_{r-1}}(g_r)_{k_r}\Phi_{rst};
\end{aligned}
$$

$$
\begin{aligned}
&F_4 : B_4 \longrightarrow K_4\\
&[g_1^{i_1}\cdots g_n^{i_n}, g_1^{j_1}\cdots g_n^{j_n}, g_1^{k_1}\cdots g_n^{k_n}, g_1^{l_1}\cdots g_n^{l_n}]\\
&\quad\mapsto \sum_{r=1}^{n} \left[\frac{k_r+l_r}{m_r}\right]\left[\frac{i_r+j_r}{m_r}\right] g_1^{i_1+j_1+k_1+l_1}\cdots g_{r-1}^{i_{r-1}+j_{r-1}+k_{r-1}+l_{r-1}}\Phi_{r^4}\\
&\qquad + \sum_{1\leqslant r<s\leqslant n} \left[\frac{k_r+l_r}{m_r}\right] g_1^{k_1+l_1}\cdots g_{r-1}^{k_{r-1}+l_{r-1}}\\
&\qquad \times \left[\frac{i_s+j_s}{m_s}\right] g_1^{i_1+j_1}\cdots g_{s-1}^{i_{s-1}+j_{s-1}}\Phi_{r^2s^2}\\
&\qquad - \sum_{1\leqslant r<s\leqslant n} \left[\frac{j_s+k_s}{m_s}\right] g_1^{j_1+k_1}\cdots g_{s-1}^{j_{s-1}+k_{s-1}} g_1^{l_1}\cdots g_{r-1}^{l_{r-1}}(g_r)_{l_r}\\
&\qquad \times g_1^{i_1}\cdots g_{s-1}^{i_{s-1}}(g_s)_{i_s}\Phi_{rs^3}\\
&\qquad - \sum_{1\leqslant r<s\leqslant n} \left[\frac{k_r+l_r}{m_r}\right] g_1^{k_1+l_1}\cdots g_{r-1}^{k_{r-1}+l_{r-1}} g_1^{j_1}\cdots g_{r-1}^{j_{r-1}}(g_r)_{j_r}\\
&\qquad \times g_1^{i_1}\cdots g_{s-1}^{i_{s-1}}(g_s)_{i_s}\Phi_{r^3s}\\
&\qquad - \sum_{1\leqslant r<s<t\leqslant n} \left[\frac{k_r+l_r}{m_r}\right] g_1^{k_1+l_1}\cdots g_{r-1}^{k_{r-1}+l_{r-1}} g_1^{j_1}\cdots g_{s-1}^{j_{s-1}}(g_s)_{j_s}\\
&\qquad \times g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}(g_t)_{i_t}\Phi_{r^2st}\\
&\qquad - \sum_{1\leqslant r<s<t\leqslant n} \left[\frac{j_s+k_s}{m_s}\right] g_1^{j_1+k_1}\cdots g_{s-1}^{j_{s-1}+k_{s-1}} g_1^{l_1}\cdots g_{r-1}^{l_{r-1}}(g_r)_{l_r}\\
&\qquad \times g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}(g_t)_{i_t}\Phi_{rs^2t}\\
&\qquad - \sum_{1\leqslant r<s<t\leqslant n} \left[\frac{i_t+j_t}{m_t}\right] g_1^{i_1+j_1}\cdots g_{t-1}^{i_{t-1}+j_{t-1}} g_1^{l_1}\cdots g_{r-1}^{l_{r-1}}(g_r)_{l_r}\\
&\qquad \times g_1^{k_1}\cdots g_{s-1}^{k_{s-1}}(g_s)_{k_s}\Phi_{rst^2}
\end{aligned}
$$

$$+\sum_{1\leqslant r<s<t<u\leqslant n} g_1^{l_1}\cdots g_{r-1}^{l_{r-1}}(g_r)_{l_r} g_1^{k_1}\cdots g_{s-1}^{k_{s-1}}(g_s)_{k_s} g_1^{j_1}\cdots g_{t-1}^{j_{t-1}}(g_t)_{j_t} \times g_1^{i_1}\cdots g_{u-1}^{i_{u-1}}(g_u)_{i_u}\Phi_{rstu}$$

for $0\leqslant i_r, j_r, k_r, l_r < m_r$ and $1\leqslant r\leqslant n$.

In general, let $\alpha := (\alpha_{11},\ldots,\alpha_{1n},\ldots,\alpha_{k1},\ldots,\alpha_{kn})$ where each $\alpha_{ij}\in[0,m_j)$ and is viewed as an integer modulo m_j for all $1\leqslant i\leqslant k$. We also write $\alpha=(\alpha_1,\ldots,\alpha_k)$ where $\alpha_u=(\alpha_{u1},\ldots,\alpha_{un})$ for $1\leqslant u\leqslant k$. For brevity, in the following we denote the group element $g_1^{\alpha_{i1}}\cdots g_n^{\alpha_{in}}$ by g^{α_i}. Given any $1\leqslant r\leqslant n$, $[a,b]\subseteq[1,k]$, $a,b\in\mathbb{N}$ and α, denote

$$\xi^{\alpha}_{r,[a,b]} := \begin{cases} \left[\dfrac{\alpha_{br}+\alpha_{b-1,r}}{m_r}\right]\cdots\left[\frac{\alpha_{a+1,r}+\alpha_{ar}}{m_r}\right] \\ \quad g_1^{\alpha_{b1}+\cdots+\alpha_{a1}}\cdots g_{r-1}^{\alpha_{b,r-1}+\cdots+\alpha_{a,r-1}}, & a-b \text{ odd};\\ \left[\dfrac{\alpha_{br}+\alpha_{b-1,r}}{m_r}\right]\cdots\left[\frac{\alpha_{a+2,r}+\alpha_{a+1,r}}{m_r}\right] \\ \quad g_1^{\alpha_{b1}+\cdots+\alpha_{a,1}}\cdots g_{r-1}^{\alpha_{b,r-1}+\cdots+\alpha_{a,r-1}}(g_r)_{\alpha_{ar}}, & a-b \text{ even}. \end{cases}$$

Define

$$\begin{aligned} &F_k\colon B_k \longrightarrow K_k \\ &[g^{\alpha_1},\ldots,g^{\alpha_k}]\mapsto \sum_{l=1}^{k}\sum_{\substack{1\leqslant r_1<\cdots<r_l\leqslant n\\ \lambda_1+\cdots+\lambda_l=k\\ \lambda_i\geqslant 1 \text{ for } 1\leqslant i\leqslant l}} (-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j}\xi^{\alpha}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\Phi_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} \end{aligned} \tag{2.4}$$

where $a_u=\sum_{i=u+1}^{l}\lambda_i+1$ and $b_u=\sum_{i=u}^{l}\lambda_i$ for $1\leqslant u\leqslant l$. Clearly, the interval $[1,k]$ is the disjoint union of the $[a_i,b_i]$'s.

Proposition 2.4. *The following diagram is commutative.*

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & B_3 & \xrightarrow{\partial_3} & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\partial_1} & B_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0\\ & & \downarrow F_3 & & \downarrow F_2 & & \downarrow F_1 & & \| & & \| & & \\ \cdots & \longrightarrow & K_3 & \xrightarrow{d} & K_2 & \xrightarrow{d} & K_1 & \xrightarrow{d} & K_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

Proof. We start with some conventions. Denote

$$\mathcal{E}_{r^{\lambda}} := \begin{cases} N_r, & \lambda \text{ even};\\ T_r, & \lambda \text{ odd}.\end{cases}$$

Then

$$\begin{aligned} d\Phi_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} &= \mathcal{E}_{r_1^{\lambda_1}}\Phi_{r_1^{\lambda_1-1}\cdots r_l^{\lambda_l}} + (-1)^{\lambda_1}\mathcal{E}_{r_2^{\lambda_2}}\Phi_{r_1^{\lambda_1}r_2^{\lambda_2-1}\cdots r_l^{\lambda_l}}+\cdots\\ &\quad + (-1)^{\lambda_1+\cdots+\lambda_{l-1}}\mathcal{E}_{r_l^{\lambda_l}}\Phi_{r_1^{\lambda_1}\cdots r_l^{\lambda_l-1}}. \end{aligned}$$

For any given $\alpha=(\alpha_{11},\ldots,\alpha_{1n},\ldots,\alpha_{k1},\ldots,\alpha_{kn})$, let $\alpha'=(\alpha_2,\ldots,\alpha_k)$, $\alpha''=(\alpha_1,\ldots,\alpha_{k-1})$ and $\alpha'_u=(\alpha_1,\ldots,\alpha_{u-1},\alpha_u+\alpha_{u+1},\alpha_{u+2},\ldots,\alpha_k)$, $\forall 1\leqslant u\leqslant k-1$.

With the above notations, $\partial_k([g^{\alpha_1},\dots,g^{\alpha_k}])$ becomes

$$g^{\alpha_1}[g^{\alpha_2},\dots,g^{\alpha_k}]+\sum_{u=1}^{k-1}(-1)^u[g^{\alpha_1},\dots,g^{\alpha_{u-1}},g^{\alpha_u+\alpha_{u+1}},g^{\alpha_{u+2}},\dots,g^{\alpha_k}]$$
$$+(-1)^k[g^{\alpha_1},\dots,g^{\alpha_{k-1}}].$$

Then the coefficient of $\Phi_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}$ in $F_{k-1}\partial_k([g^{\alpha_1},\dots,g^{\alpha_k}])$ is

$$(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j}\left(g^{\alpha_1}\xi^{\alpha'}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha'}_{r_l,[a_l,b_l]}+\sum_{u=1}^{k-1}(-1)^u\xi^{\alpha'_u}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha'_u}_{r_l,[a_l,b_l]}\right.$$
$$\left.+(-1)^k\xi^{\alpha''}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha''}_{r_l,[a_l,b_l]}\right) \tag{2.5}$$

where $a_u=\sum_{i=u+1}^{l}\lambda_i+1$ and $b_u=\sum_{i=u}^{l}\lambda_i$. For $1\leqslant r\leqslant n$, $[a,b]\subseteq[1,k-1]$, $a,b\in\mathbb{N}$ and α, we have

$$\xi^{\alpha'}_{r,[a,b]}=\xi^{\alpha}_{r,[a+1,b+1]},\quad \xi^{\alpha''}_{r,[a,b]}=\xi^{\alpha}_{r,[a,b]}\quad\text{and}\quad \xi^{\alpha'_u}_{r,[a,b]}=\begin{cases}\xi^{\alpha}_{r,[a+1,b+1]}, & \text{if } u<a;\\ \xi^{\alpha}_{r,[a,b]}, & \text{if } u>b.\end{cases}$$

Hence

$$\begin{aligned}&\sum_{u=1}^{k-1}(-1)^u\xi^{\alpha'_u}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha'_u}_{r_l,[a_l,b_l]}\\
&\quad=\sum_{i=1}^{l}\sum_{u=a_i}^{b_i}(-1)^u\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_{i-1},[a_{i-1}+1,b_{i-1}+1]}\xi^{\alpha'_u}_{r_i,[a_i,b_i]}\\
&\qquad\times\xi^{\alpha}_{r_{i+1},[a_{i+1},b_{i+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\\
&\quad=\sum_{i=1}^{l}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_{i-1},[a_{i-1}+1,b_{i-1}+1]}\xi^{\alpha}_{r_{i+1},[a_{i+1},b_{i+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\\
&\qquad\times\sum_{u=a_i}^{b_i}(-1)^u\xi^{\alpha'_u}_{r_i,[a_i,b_i]}.\end{aligned}$$

Therefore we can rewrite (2.5) as

$$(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j}\left(g^{\alpha_1}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_l,[a_l+1,b_l+1]}+(-1)^k\xi^{\alpha}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\right.$$
$$+\sum_{i=1}^{l}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_{i-1},[a_{i-1}+1,b_{i-1}+1]}\xi^{\alpha}_{r_{i+1},[a_{i+1},b_{i+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]} \tag{2.6}$$
$$\left.\times\sum_{u=a_i}^{b_i}(-1)^u\,\xi^{\alpha'_u}_{r_i,[a_i,b_i]}\right).$$

It remains to compute $\sum_{u=a_i}^{b_i}(-1)^u\xi_{r_i,[a_i,b_i]}^{\alpha'_u}$. This is split into two cases according to the parity of b_i-a_i.

If b_i-a_i is odd, then

$$\begin{aligned}
&\sum_{u=a_i}^{b_i}(-1)^u\xi_{r_i,[a_i,b_i]}^{\alpha'_u} \qquad (2.7)\\
&=\Bigg((-1)^{a_i}\left[\frac{\alpha_{b_i+1,r_i}+\alpha_{b_i,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+2,r_i}+(\alpha_{a_i,r_i}+\alpha_{a_i+1,r_i})'}{m_{r_i}}\right]\\
&\quad+(-1)^{a_i+1}\left[\frac{\alpha_{b_i+1,r_i}+\alpha_{b_i,r_i}}{m_{r_i}}\right]\cdots\left[\frac{(\alpha_{a_i+1,r_i}+\alpha_{a_i+2,r_i})'+\alpha_{a_i,r_i}}{m_{r_i}}\right]\\
&\quad+\cdots+(-1)^{b_i}\left[\frac{(\alpha_{b_i,r_i}+\alpha_{b_i+1,r_i})'+\alpha_{b_i-1,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+1,r_i}+\alpha_{a_i,r_i}}{m_{r_i}}\right]\Bigg)\\
&\quad\cdot g_1^{\alpha_{b_i+1,1}+\cdots+\alpha_{a_i,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}+\cdots+\alpha_{a_i,r_i-1}}\\
&\quad\overset{Lemma 2.3}{=}\Bigg((-1)^{a_i}\left[\frac{\alpha_{b_i+1,r_i}+\alpha_{b_i,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+2,r_i}+\alpha_{a_i+1,r_i}}{m_{r_i}}\right]\\
&\quad+(-1)^{b_i}\left[\frac{\alpha_{b_i,r_i}+\alpha_{b_i-1,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+1,r_i}+\alpha_{a_i,r_i}}{m_{r_i}}\right]\Bigg)\\
&\quad\cdot g_1^{\alpha_{b_i+1,1}+\cdots+\alpha_{a_i,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}+\cdots+\alpha_{a_i,r_i-1}}\\
&=(-1)^{b_i}\xi_{r_i,[a_i,b_i]}^{\alpha}g_1^{\alpha_{b_i+1,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}}+(-1)^{a_i}\xi_{r_i,[a_i+1,b_i+1]}^{\alpha}g_1^{\alpha_{a_i,1}}\cdots g_{r_i-1}^{\alpha_{a_i,r_i-1}}.
\end{aligned}$$

If b_i-a_i is even, then similarly we have

$$\begin{aligned}
&\sum_{u=a_i}^{b_i}(-1)^u\xi_{r_i,[a_i,b_i]}^{\alpha'_u} \qquad (2.8)\\
&=(-1)^{a_i}\left[\frac{\alpha_{b_i+1,r_i}+\alpha_{b_i,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+3,r_i}+\alpha_{a_i+2,r_i}}{m_{r_i}}\right]\\
&\quad g_1^{\alpha_{b_i+1,1}+\alpha_{a_i,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}+\cdots+\alpha_{a_i,r_i-1}}(g_{r_i})_{(\alpha_{a_i,r_i}+\alpha_{a_i+1,r_i})'}\\
&\quad+\Bigg((-1)^{a_i+1}\left[\frac{\alpha_{b_i+1,r_i}+\alpha_{b_i,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+3,r_i}+(\alpha_{a_i+1,r_i}+\alpha_{a_i+2,r_i})'}{m_{r_i}}\right]\\
&\quad+(-1)^{a_i+2}\left[\frac{\alpha_{b_i+1,r_i}+\alpha_{b_i,r_i}}{m_{r_i}}\right]\cdots\left[\frac{(\alpha_{a_i+2,r_i}+\alpha_{a_i+3,r_i})'+\alpha_{a_i+1,r_i}}{m_{r_i}}\right]+\cdots\\
&\quad+(-1)^{b_i}\left[\frac{(\alpha_{b_i,r_i}+\alpha_{b_i+1,r_i})'+\alpha_{b_i-1,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+2,r_i}+\alpha_{a_i+1,r_i}}{m_{r_i}}\right]\Bigg)\\
&\quad\cdot g_1^{\alpha_{b_i+1,1}+\cdots+\alpha_{a_i,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}+\cdots+\alpha_{a_i,r_i-1}}(g_{r_i})_{\alpha_{a_i,r_i}}\\
&=(-1)^{a_i}\left[\frac{\alpha_{b_i+1,r_i}+\alpha_{b_i,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+3,r_i}+\alpha_{a_i+2,r_i}}{m_{r_i}}\right]\\
&\quad g_1^{\alpha_{b_i+1,1}+\alpha_{a_i,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}+\cdots+\alpha_{a_i,r_i-1}}
\end{aligned}$$

$$\cdot\left((g_{r_i})_{\alpha_{a_i,r_i}+\alpha_{a_i+1,r_i}} - \left[\frac{\alpha_{a_i,r_i}+\alpha_{a_i+1,r_i}}{m_{r_i}}\right] N_{r_i}\right)$$
$$+\left((-1)^{a_i+1}\left[\frac{\alpha_{b_i+1,r_i}+\alpha_{b_i,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+3,r_i}+\alpha_{a_i+2,r_i}}{m_{r_i}}\right]\right.$$
$$\left.+(-1)^{b_i}[\frac{\alpha_{b_i,r_i}+\alpha_{b_i-1,r_i}}{m_{r_i}}]\cdots\left[\frac{\alpha_{a_i+2,r_i}+\alpha_{a_i+1,r_i}}{m_{r_i}}\right]\right)$$
$$\cdot g_1^{\alpha_{b_i+1,1}+\cdots+\alpha_{a_i,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}+\cdots+\alpha_{a_i,r_i-1}}(g_{r_i})_{\alpha_{a_i,r_i}}$$
$$=(-1)^{a_i}\left[\frac{\alpha_{b_i+1,r_i}+\alpha_{b_i,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+3,r_i}+\alpha_{a_i+2,r_i}}{m_{r_i}}\right]$$
$$g_1^{\alpha_{b_i+1,1}+\alpha_{a_i,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}+\cdots+\alpha_{a_i,r_i-1}}$$
$$\cdot\left((g_{r_i})_{\alpha_{a_i,r_i}+\alpha_{a_i+1,r_i}}-(g_{r_i})_{\alpha_{a_i,r_i}}-\left[\frac{\alpha_{a_i,r_i}+\alpha_{a_i+1,r_i}}{m_{r_i}}\right]N_{r_i}\right)$$
$$+(-1)^{b_i}\left[\frac{\alpha_{b_i,r_i}+\alpha_{b_i-1,r_i}}{m_{r_i}}\right]\cdots\left[\frac{\alpha_{a_i+2,r_i}+\alpha_{a_i+1,r_i}}{m_{r_i}}\right]$$
$$g_1^{\alpha_{b_i+1,1}+\cdots+\alpha_{a_i,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}+\cdots+\alpha_{a_i,r_i-1}}(g_{r_i})_{\alpha_{a_i,r_i}}$$
$$=(-1)^{a_i}\xi^{\alpha}_{r_i,[a_i+1,b_i+1]}g_1^{\alpha_{a_i,1}}\cdots g_{r_i}^{\alpha_{a_i,r_i}}+(-1)^{a_i+1}\xi^{\alpha}_{r_i,[a_i,b_i+1]}\mathcal{E}_{{r_i}^{b_i-a_i}}$$
$$+(-1)^{b_i}\xi^{\alpha}_{r_i,[a_i,b_i]}g_1^{\alpha_{b_i+1,1}}\cdots g_{r_i-1}^{\alpha_{b_i+1,r_i-1}}.$$

On the other hand, the term $\Phi_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}$ in $dF_k([g^{\alpha_1},\ldots,g^{\alpha_k}])$ comes from the differential of the terms

$$\Phi_{r_1^{\lambda_1+1}r_2^{\lambda_2}\cdots r_l^{\lambda_l}},\ \cdots,\ \Phi_{r_1^{\lambda_1}\cdots r_{l-1}^{\lambda_{l-1}}r_l^{\lambda_l+1}},\ \ \Phi_{sr_1^{\lambda_1}\cdots r_l^{\lambda_l}},\ \ \Phi_{r_1^{\lambda_1}sr_2^{\lambda_2}\cdots r_l^{\lambda_l}},\ \cdots,$$
$$\Phi_{r_1^{\lambda_1}\cdots r_{l-1}^{\lambda_{l-1}}sr_l^{\lambda_l}},\ \ \Phi_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}s}$$

in $F_k([g^{\alpha_1},\ldots,g^{\alpha_k}])$. Therefore, its coefficient is

$$\sum_{1\leqslant u\leqslant l}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{j\neq u}\lambda_j}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_{u-1},[a_{u-1}+1,b_{u-1}+1]}\xi^{\alpha}_{r_u,[a_u,b_u+1]} \tag{2.9}$$
$$\cdot\xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}(-1)^{\lambda_1+\cdots+\lambda_{u-1}}\mathcal{E}_{r_u^{\lambda_u+1}}$$
$$+\sum_{s=1}^{r_1-1}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\xi^{\alpha}_{s,[k,k]}T_s$$
$$+\sum_{u=1}^{l-1}\sum_{s=r_u+1}^{r_{u+1}-1}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_u,[a_u+1,b_u+1]}$$

$$\cdot\, \xi_{s,[a_u,a_u]}\xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}(-1)^{\lambda_1+\cdots+\lambda_u}T_s$$
$$+\sum_{s=r_l+1}^{n}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_l,[a_l+1,b_l+1]}$$
$$\times\, \xi^{\alpha}_{s,[1,1]}(-1)^{\sum_{i=1}^{l}\lambda_i}T_s.$$

Noting that $\xi^{\alpha}_{s,[a,a]}T_s = g_1^{\alpha_{a1}}\cdots g_{s-1}^{\alpha_{a,s-1}}(g_s^{\alpha_{a,s}}-1)$, then one has the following equations:

$$\sum_{s=1}^{r_1-1}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\xi^{\alpha}_{s,[k,k]}T_s$$
$$=(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\sum_{s=1}^{r_1-1}\xi^{\alpha}_{s,[k,k]}T_s$$
$$=(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}(g_1^{\alpha_{k1}}\cdots g_{r_1-1}^{\alpha_{k,r_1-1}}-1),$$

$$\sum_{u=1}^{l-1}\sum_{s=r_u+1}^{r_{u+1}-1}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_u,[a_u+1,b_u+1]}$$
$$\cdot\, \xi_{s,[a_u,a_u]}\xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}(-1)^{\lambda_1+\cdots+\lambda_u}T_s$$
$$=\sum_{u=1}^{l-1}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=u+1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_u,[a_u+1,b_u+1]}$$
$$\cdot\, \xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\sum_{s=r_u+1}^{r_{u+1}-1}\xi_{s,[a_u,a_u]}T_s$$
$$=\sum_{u=1}^{l-1}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=u+1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_u,[a_u+1,b_u+1]}$$
$$\cdot\, \xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}(g_1^{\alpha_{a_u1}}\cdots g_{r_{u+1}-1}^{\alpha_{a_u,r_{u+1}-1}}-g_1^{\alpha_{a_u1}}\cdots g_{r_u}^{\alpha_{a_u,r_u}}),$$

$$\sum_{s=r_l+1}^{n}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_l,[a_l+1,b_l+1]}\xi^{\alpha}_{s,[1,1]}(-1)^{\sum_{i=1}^{l}\lambda_i}T_s$$
$$=(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_l,[a_l+1,b_l+1]}(g_1^{\alpha_{11}}\cdots g_n^{\alpha_{1n}}-g_1^{\alpha_{11}}\cdots g_{r_l}^{\alpha_{1,r_l}}).$$

With these, (2.9) becomes

$$\sum_{1\leqslant u\leqslant l}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{j=u+1}^{l}\lambda_j}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_{u-1},[a_{u-1}+1,b_{u-1}+1]} \tag{2.10}$$
$$\cdot\, \xi^{\alpha}_{r_u,[a_u,b_u+1]}\xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\mathcal{E}_{r_u^{\lambda_u+1}}$$

$$
\begin{aligned}
&+(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}(g_1^{\alpha_{k1}}\cdots g_{r_1-1}^{\alpha_{k,r_1-1}}-1)\\
&+\sum_{u=1}^{l-1}(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j+\sum_{i=u+1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_u,[a_u+1,b_u+1]}\\
&\cdot\xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}(g_1^{\alpha_{a_u1}}\cdots g_{r_{u+1}-1}^{\alpha_{a_u,r_{u+1}-1}}-g_1^{\alpha_{a_u1}}\cdots g_{r_u}^{\alpha_{a_u,r_u}})\\
&+(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_l,[a_l+1,b_l+1]}(g_1^{\alpha_{11}}\cdots g_n^{\alpha_{1n}}-g_1^{\alpha_{11}}\cdots g_{r_l}^{\alpha_{1,r_l}}).
\end{aligned}
$$

We need to prove that the two formulas (2.6) and (2.10) are equal. By cancelling their obvious common terms, namely the first two terms of (2.6), it suffices to prove

$$
\begin{aligned}
&\sum_{i=1}^{l}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_{i-1},[a_{i-1}+1,b_{i-1}+1]}\xi^{\alpha}_{r_{i+1},[a_{i+1},b_{i+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\\
&\quad\times\sum_{u=a_i}^{b_i}(-1)^u\xi^{\alpha'_u}_{r_i,[a_i,b_i]}\\
&\qquad=\sum_{u=1}^{l}(-1)^{\sum_{i=u+1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_{u-1},[a_{u-1}+1,b_{u-1}+1]}\xi^{\alpha}_{r_u,[a_u,b_u+1]}\\
&\qquad\quad\times\xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\mathcal{E}_{r_u^{\lambda_u+1}}\\
&\qquad\quad+\sum_{u=1}^{l-1}(-1)^{\sum_{i=u+1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_u,[a_u+1,b_u+1]}\\
&\qquad\quad\times\xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\cdot(g_1^{\alpha_{a_u1}}\cdots g_{r_{u+1}-1}^{\alpha_{a_u,r_{u+1}-1}}-g_1^{\alpha_{a_u1}}\cdots g_{r_u}^{\alpha_{a_u,r_u}})\\
&\qquad\quad+(-1)^{\sum_{i=1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1,b_1]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}g_1^{\alpha_{k1}}\cdots g_{r_1-1}^{\alpha_{k,r_1-1}}\\
&\qquad\quad-\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_l,[a_l+1,b_l+1]}g_1^{\alpha_{11}}\cdots g_{r_l}^{\alpha_{1,r_l}}.
\end{aligned}
$$

Note that the latter is equal to

$$
\begin{aligned}
&\sum_{u=1}^{l}(-1)^{\sum_{i=u+1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_{u-1},[a_{u-1}+1,b_{u-1}+1]}\xi^{\alpha}_{r_u,[a_u,b_u+1]}\\
&\quad\xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}\mathcal{E}_{r_u^{\lambda_u+1}}+\sum_{u=1}^{l}(-1)^{\sum_{i=u}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\\
&\quad\xi^{\alpha}_{r_{u-1},[a_{u-1}+1,b_{u-1}+1]}\xi^{\alpha}_{r_u,[a_u,b_u]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}g_1^{\alpha_{a_{u-1}1}}\cdots g_{r_u-1}^{\alpha_{a_{u-1},r_u-1}}\\
&\quad-\sum_{u=1}^{l}(-1)^{\sum_{i=u+1}^{l}\lambda_i}\xi^{\alpha}_{r_1,[a_1+1,b_1+1]}\cdots\xi^{\alpha}_{r_u,[a_u+1,b_u+1]}\\
&\quad\xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]}\cdots\xi^{\alpha}_{r_l,[a_l,b_l]}g_1^{\alpha_{a_u1}}\cdots g_{r_u}^{\alpha_{a_u,r_u}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{u=1}^{l} \xi^{\alpha}_{r_1,[a_1+1,b_1+1]} \cdots \xi^{\alpha}_{r_{u-1},[a_{u-1}+1,b_{u-1}+1]} \xi^{\alpha}_{r_{u+1},[a_{u+1},b_{u+1}]} \cdots \xi^{\alpha}_{r_l,[a_l,b_l]} \\
&\cdot \Big((-1)^{\sum_{i=u+1}^{l} \lambda_i} \xi^{\alpha}_{r_u,[a_u,b_u+1]} \mathcal{E}_{r_u^{\lambda_u+1}} + (-1)^{\sum_{i=u}^{l} \lambda_i} \xi^{\alpha}_{r_u,[a_u,b_u]} g_1^{\alpha_{a_{u-1}1}} \cdots g_{r_u-1}^{\alpha_{a_{u-1},r_u-1}} \\
&- (-1)^{\sum_{i=u+1}^{l} \lambda_i} \xi^{\alpha}_{r_u,[a_u+1,b_u+1]} g_1^{\alpha_{a_u1}} \cdots g_{r_u}^{\alpha_{a_u,r_u}} \Big).
\end{aligned}
$$

Now it is enough to verify that

$$
\begin{aligned}
\sum_{u=a_i}^{b_i} (-1)^u \xi^{\alpha'_u}_{r_i,[a_i,b_i]} &= (-1)^{\sum_{i=u+1}^{l} \lambda_i} \xi^{\alpha}_{r_u,[a_u,b_u+1]} \mathcal{E}_{r_u^{\lambda_u+1}} \\
&\quad + (-1)^{\sum_{i=u}^{l} \lambda_i} \xi^{\alpha}_{r_u,[a_u,b_u]} g_1^{\alpha_{a_{u-1}1}} \cdots g_{r_u-1}^{\alpha_{a_{u-1},r_u-1}} \\
&\quad - (-1)^{\sum_{i=u+1}^{l} \lambda_i} \xi^{\alpha}_{r_u,[a_u+1,b_u+1]} g_1^{\alpha_{a_u1}} \cdots g_{r_u}^{\alpha_{a_u,r_u}}.
\end{aligned} \tag{2.11}
$$

The verification is split into two cases. If $b_i - a_i$ is even, then the equality is immediate simply by noting that

$$
a_u = \sum_{i=u+1}^{l} \lambda_i + 1, \quad b_u = \sum_{i=u}^{l} \lambda_i.
$$

If $b_i - a_i$ is odd, noting that

$$
\begin{aligned}
&(-1)^{\sum_{i=u+1}^{l} \lambda_i} \xi^{\alpha}_{r_u,[a_u,b_u+1]} \mathcal{E}_{r_u^{\lambda_u+1}} \\
&\quad = (-1)^{\sum_{i=u+1}^{l} \lambda_i} \xi^{\alpha}_{r_u,[a_u+1,b_u+1]} g_1^{\alpha_{a_u,1}} \cdots g_{r_u-1}^{\alpha_{a_u,r_u-1}} (g_{r_u})_{\alpha_{a_u,r_u}} (g_{r_u} - 1) \\
&\quad = (-1)^{\sum_{i=u+1}^{l} \lambda_i} \xi^{\alpha}_{r_u,[a_u+1,b_u+1]} g_1^{\alpha_{a_u,1}} \cdots g_{r_u-1}^{\alpha_{a_u,r_u-1}} (g_{r_u}^{\alpha_{a_u,r_u}} - 1),
\end{aligned}
$$

then the equality (2.11) follows.

The proof is completed. □

2.3. Normalized cocycles

Denote

$$
\eta^{\alpha}_{r,[a,b]} := \begin{cases} \left[\dfrac{\alpha_{br} + \alpha_{b-1,r}}{m_r}\right] \cdots \left[\dfrac{\alpha_{a+1,r} + \alpha_{ar}}{m_r}\right], & b-a \text{ odd}; \\ \left[\dfrac{\alpha_{br} + \alpha_{b-1,r}}{m_r}\right] \cdots \left[\dfrac{\alpha_{a+2,r} + \alpha_{a+1,r}}{m_r}\right] \alpha_{ar}, & b-a \text{ even}. \end{cases}
$$

Corollary 2.5. *The following k-cochains $\omega \in \mathrm{Hom}_{\mathbb{Z}G}(B_k, \mathbb{k}^*)$ given by*

$$
\omega([g^{\alpha_1}, \ldots, g^{\alpha_k}]) = \prod_{l=1}^{k} \prod_{\substack{1 \leqslant r_1 < \cdots < r_l \leqslant n \\ \lambda_1 + \cdots + \lambda_l = k, \lambda_1 \text{ odd} \\ \lambda_i \geqslant 1 \text{ for } 1 \leqslant i \leqslant l}} \zeta_{m_{r_1}}^{(-1)^{\sum_{1 \leqslant i < j \leqslant l} \lambda_i \lambda_j} \eta^{\alpha}_{r_1,[a_1,b_1]} \cdots \eta^{\alpha}_{r_l,[a_l,b_l]} a_{r_1^{\lambda_1} \cdots r_l^{\lambda_l}}} \tag{2.12}
$$

where $a_u = \sum_{i=u+1}^{l} \lambda_i + 1$, $b_u = \sum_{i=u}^{l} \lambda_i$ *and* $0 \leqslant a_{r_1^{\lambda_1} \cdots r_l^{\lambda_l}} < m_{r_1}$ *for* $1 \leqslant r_1 < \cdots < r_l \leqslant n$ *form a complete set of representatives of* k*-cocycles of the complex* $(B_\bullet^*, \partial_\bullet^*)$.

Proof. It follows from the chain map (2.4) and theorem 2.1. □

2.4. A chain map from $(K_\bullet, d_\bullet)$ to $(B_\bullet, \partial_\bullet)$

For completeness, we also include a chain map from the Koszul-like resolution $(K_\bullet, d_\bullet)$ to the normalized bar resolution $(B_\bullet, \partial_\bullet)$. This chain map is very useful for comparing cohomology classes of normalized cocycles and for studying the whole cohomology ring structure, etc.

Denote an ordered set of λ elements as

$$\Lambda_{r^\lambda} := \begin{cases} (N_r, g_r, N_r, g_r, \ldots, N_r, g_r), & \lambda \text{ even;} \\ (g_r, N_r, g_r, N_r, g_r \ldots, N_r, g_r), & \lambda \text{ odd.} \end{cases}$$

Given a set of positive integers λ_1, $\lambda_2, \ldots, \lambda_l$ with $\lambda_1 + \cdots + \lambda_l = k$, let $\text{Shuffle}(\lambda_1, \ldots, \lambda_l)$ be the subset of the permutation group S_k such that the elements of it preserve the order of elements of each block of the partition $(\lambda_1, \ldots, \lambda_l)$. For each k, define a map

$$\begin{aligned} G_k \colon K_k &\longrightarrow B_k \\ \Phi_{r_1^{\lambda_1} \cdots r_l^{\lambda_l}} &\mapsto \sum_{\sigma \in \text{Shuffle}(\lambda_1, \ldots, \lambda_l)} (-1)^\sigma [\sigma(\Lambda_{r_1^{\lambda_1}}, \ldots, \Lambda_{r_l^{\lambda_l}})]. \end{aligned}$$

Proposition 2.6. *We have the following commutative diagram*

$$\begin{tikzcd} \cdots \arrow[r] & K_3 \arrow[r, "d"] \arrow[d, "G_3"] & K_2 \arrow[r, "d"] \arrow[d, "G_2"] & K_1 \arrow[r, "d"] \arrow[d, "G_1"] & K_0 \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ \cdots \arrow[r] & B_3 \arrow[r, "\partial_3"] & B_2 \arrow[r, "\partial_2"] & B_1 \arrow[r, "\partial_1"] & B_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Proof. By direct verification similarly as the proof of proposition 2.4. The detail is omitted. □

2.5. A translation to quantum field theory

Now we follow the notations in [**29**] and translate our result into quantum field theory language. Let $G = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_n}$ where $N_i | N_{i+1}$ for $1 \leqslant i \leqslant n-1$. Let $k = d + 1$ be the spacetime dimension. For $1 \leqslant l \leqslant d+1$, $1 \leqslant r_1 < \cdots < r_l \leqslant n$, $\lambda_i \geqslant 1$ for $1 \leqslant i \leqslant l$, define

$$\phi_{r_i^{\lambda_i}} = \begin{cases} A_{r_i} dA_{r_i} \cdots dA_{r_i}, & \text{if } \lambda_i \text{ odd;} \\ dA_{r_i} \cdots dA_{r_i}, & \text{if } \lambda_i \text{ even.} \end{cases}$$

We generalize the correspondence between the partition functions of fields and cocycles given in [**29**].

The generalized correspondence connects the part

$$\zeta_{N_{r_1}}^{(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j}\eta^{\alpha}_{r_1,[a_1,b_1]}\cdots\eta^{\alpha}_{r_l,[a_l,b_l]}a_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}}$$

of $(d+1)$-cocycle ω_{d+1} and the partition function

$$\zeta_{N_{r_1}}^{(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j}a_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}}\frac{N_{r_1}^{\lambda_1-2[\lambda_1/2]}\cdots N_{r_l}^{\lambda_l-2[\lambda_l/2]}}{(2\pi)^{[(\lambda_1+1)/2]+\cdots+[(\lambda_l+1)/2]}}\int\phi_{r_l^{\lambda_l}}\cdots\phi_{r_1^{\lambda_1}}$$

where the corresponding terms of A_u and dA_u are given in [**29**] and the order of A_u and dA_u is so arranged that their positions indicate which component of α they correspond to. Note that $\alpha=(\alpha_1,\ldots,\alpha_{d+1})$, $\lambda_1+\cdots+\lambda_l=d+1$, and λ_1 is odd. Our result reveals the fact that we do not need higher form fields B, C, etc, to get a complete set of representatives of cocycles.

Now we explain how we get these partition functions. First, any 1-form field is the linear combination of the wedge products of some A_u and dA_u where each A_u appears at most once, i.e. the linear combination of $\phi_{r_l^{\lambda_l}}\cdots\phi_{r_1^{\lambda_1}}$ for some $1\leqslant l\leqslant d+1$, $1\leqslant r_1<\cdots<r_l\leqslant n$, $\lambda_i\geqslant 1$ for $1\leqslant i\leqslant l$. After integration by part on $\int\phi_{r_l^{\lambda_l}}\cdots\phi_{r_1^{\lambda_1}}$, we need only consider those terms with λ_1 odd.

Due to a discrete $\mathbb{Z}_N$ gauge symmetry, and the gauge transformation must be identified by 2π, we have the following general rules:

$$\oint A_u=\frac{2\pi n_u}{N_u}\mod 2\pi,\qquad \oint\delta A_u=0\mod 2\pi.$$

We consider a spacetime with a volume size L^{d+1} where L is the length of one dimension, for example T^{d+1} torus. The allowed large gauge transformation implies that locally A can be:

$$A_{u,\mu}=\frac{2\pi n_u\,\mathrm{d}x_\mu}{N_uL},\qquad \delta A_u=\frac{2\pi m_u\,\mathrm{d}x_\mu}{L}.$$

Now we consider the partition function $\exp(ik_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}\int\phi_{r_l^{\lambda_l}}\cdots\phi_{r_1^{\lambda_1}})$ with λ_1 odd. Note that $\delta(dA_u)=0$. Thus for the large gauge transformation, we have $k_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}\int\delta(\phi_{r_l^{\lambda_l}}\cdots\phi_{r_1^{\lambda_1}})=0\mod 2\pi$. This implies

$$k_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}=p_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}\frac{N_{r_2}^{\lambda_2-2[\lambda_2/2]}\cdots N_{r_l}^{\lambda_l-2[\lambda_l/2]}}{(2\pi)^{[(\lambda_1+1)/2]+\cdots+[(\lambda_l+1)/2]-1}}$$

where $p_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}\in\mathbb{Z}$.

For the flux identification, we have

$$k_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}\int \phi_{r_l^{\lambda_l}}\cdots\phi_{r_1^{\lambda_1}} = \frac{(2\pi)^{[(\lambda_1+1)/2]+\cdots+[(\lambda_l+1)/2]}n_{r_1}^{[(\lambda_1+1)/2]}\cdots n_{r_l}^{[(\lambda_l+1)/2]}}{N_{r_1}^{\lambda_1-2[\lambda_1/2]}\cdots N_{r_l}^{\lambda_l-2[\lambda_l/2]}}.$$

Hence

$$\begin{aligned}&(2\pi)^{[(\lambda_1+1)/2]+\cdots+[(\lambda_l+1)/2]-1}k_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}\\ &\quad\simeq (2\pi)^{[(\lambda_1+1)/2]+\cdots+[(\lambda_l+1)/2]-1}k_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} + N_{r_1}^{\lambda_1-2[\lambda_1/2]}\cdots N_{r_l}^{\lambda_l-2[\lambda_l/2]}.\end{aligned}$$

Here $\simeq$ means the level identification. Therefore, the cyclic period of $p_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}$ is N_{r_1}.

Finally let $(-1)^{\sum_{1\leqslant i<j\leqslant l}\lambda_i\lambda_j}p_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}} = a_{r_1^{\lambda_1}\cdots r_l^{\lambda_l}}$. Then we get the partition functions in correspondence with cocycles.

3. On Braided linear Gr-categories

The monoidal category of finite-dimensional vector spaces graded by a group G, with the usual tensor product and associativity constraint given by a 3-cocycle ω is denoted by Vec_G^ω. Such a monoidal category is called a linear Gr-category. The terminology goes back to Hoàng Xuân Sính [**14**], a student of Grothendieck. The aim of this section is to give a complete description to all braided linear Gr-categories with a help of the explicit formulas of normalized 3-cocycles. This extends the related partial results obtained in [**1**, **18**, **21**, **24**] to full generality.

3.1. Monoidal structures

Recall that the category Vec_G of finite-dimensional G-graded vector spaces has simple objects $\{S_g|g\in G\}$ where $(S_g)_h = \delta_{g,h}\Bbbk$, $\forall h\in G$. The tensor product is given by $S_g\otimes S_h = S_{gh}$, and S_1 (1 is the identity of G) is the unit object. Without loss of generality we may assume that the left and right unit constraints are identities. If a is an associativity constraint on Vec_G, then it is given by $a_{S_f,S_g,S_h} = \omega(f,g,h)\,\mathrm{id}$, where $\omega: G\times G\times G\to \Bbbk^*$ is a function. The pentagon axiom and the triangle axiom give

$$\omega(ef,g,h)\omega(e,f,gh) = \omega(e,f,g)\omega(e,fg,h)\omega(f,g,h),$$
$$\omega(f,1,g) = 1,$$

which say exactly that ω is a normalized 3-cocycle on G. Note that cohomologous cocycles define equivalent monoidal structures, therefore the classification of monoidal structures on Vec_G is equivalent to determining a complete set of representatives of normalized 3-cocycles on G.

3.2. Normalized 3-cocycles

In the special case $k = 3$, if we abbreviate a_{r^3} by a_r, a_{rs^2} by a_{rs}, then (2.12) becomes

$$\begin{aligned} &\omega :\ B_3 \longrightarrow \Bbbk^* \\ &[g_1^{i_1}\cdots g_n^{i_n}, g_1^{j_1}\cdots g_n^{j_n}, g_1^{k_1}\cdots g_n^{k_n}] \\ &\mapsto \prod_{r=1}^{n} \zeta_{m_r}^{a_r i_r[(j_r+k_r)/m_r]} \prod_{1\leqslant r<s\leqslant n} \zeta_{m_r}^{a_{rs}k_r[(i_s+j_s)/m_s]} \prod_{1\leqslant r<s<t\leqslant n} \zeta_{m_r}^{-a_{rst}k_r j_s i_t} \end{aligned} \tag{3.1}$$

where $0 \leqslant a_r, a_{rs}, a_{rst} < m_r$.

Remark 3.1. The present complete set of representatives of normalized 3-cocycles is slightly different from that in [**18**, **20**]. Of course they are equivalent up to cohomology.

3.3. Braided structures

Now we consider the braided structures on a linear Gr-category Vec_G^{ω}. Recall that a braiding in Vec_G^{ω} is a commutativity constraint $c : \otimes \to \otimes^{\mathrm{op}}$ satisfying the hexagon axiom. Note that c is given by $c_{S_x,S_y} = \mathcal{R}(x,y)\,\mathrm{id}$, where $\mathcal{R} : G\times G \to \Bbbk^*$ is a function, and the hexagon axiom of c says that

$$\frac{\mathcal{R}(xy,z)}{\mathcal{R}(x,z)\mathcal{R}(y,z)}\frac{\omega(x,z,y)}{\omega(x,y,z)\omega(z,x,y)} = 1 = \frac{\mathcal{R}(x,yz)}{\mathcal{R}(x,y)\mathcal{R}(x,z)}\frac{\omega(x,y,z)\omega(y,z,x)}{\omega(y,x,z)} \tag{3.2}$$

for all $x, y, z \in G$.

In other words, $\mathcal{R}$ is a quasi-bicharacter of G with respect to ω. Therefore, the classification of braidings in Vec_G^{ω} is equivalent to determining all the quasi-bicharacters of G with respect to ω. It is interesting to remark that the braided monoidal structures $(\omega, \mathcal{R})$ on Vec_G appeared already in the 1950s in terms of the so-called abelian cohomology of Eilenberg and Mac Lane [**6**, **7**].

3.4. Quasi-bicharacters

Clearly, any quasi-bicharacter $\mathcal{R}$ is uniquely determined by the following values:

$$r_{ij} := \mathcal{R}(g_i, g_j), \quad \text{for all } 1 \leqslant i,\ j \leqslant n.$$

Proposition 3.2. *Let $r_{ij} \in \Bbbk^*$ for $1 \leqslant i,\ j \leqslant n$. Then there is a quasi-bicharacter $\mathcal{R}$ with respect to ω satisfying $\mathcal{R}(g_i, g_j) = r_{ij}$ if and only if the following equations are satisfied:*

$$\begin{aligned} r_{ii}^{m_i} = \zeta_{m_i}^{a_i} = \zeta_{m_i}^{-a_i}, &\quad \text{for } 1 \leqslant i \leqslant n, \\ r_{ij}^{m_i} = r_{ji}^{m_i} = 1, \quad a_{ij} = 0, &\quad \text{for } 1 \leqslant i < j \leqslant n, \\ a_{rst} = 0, &\quad \text{for } 1 \leqslant r < s < t \leqslant n. \end{aligned}$$

Proof. '⇒'. For the case $r < s < t$, consider $\mathcal{R}(g_t g_s, g_r)$ and $\mathcal{R}(g_s g_t, g_r)$ which obviously are equal. By (3.2), we have

$$\begin{aligned}\mathcal{R}(g_t g_s, g_r) &= \mathcal{R}(g_t, g_r)\mathcal{R}(g_s, g_r)\frac{\omega(g_r, g_t, g_s)\omega(g_t, g_s, g_r)}{\omega(g_t, g_r, g_s)} \\ &= \mathcal{R}(g_t, g_r)\mathcal{R}(g_s, g_r)\zeta_{m_r}^{-a_{rst}},\end{aligned}$$

$$\begin{aligned}\mathcal{R}(g_s g_t, g_r) &= \mathcal{R}(g_s, g_r)\mathcal{R}(g_t, g_r)\frac{\omega(g_r, g_s, g_t)\omega(g_s, g_t, g_r)}{\omega(g_s, g_r, g_t)} \\ &= \mathcal{R}(g_s, g_r)\mathcal{R}(g_t, g_r).\end{aligned}$$

Therefore, $\zeta_{m_r}^{-a_{rst}} = 1$. Since $0 \leqslant a_{rst} < m_r$, we arrive at $a_{rst} = 0$.

For any $1 \leqslant i \leqslant n$, applying (3.2) iteratively, we have $\mathcal{R}(g_i, g_i^s) = \mathcal{R}(g_i, g_i)^s$ and $\mathcal{R}(g_i^s, g_i) = \mathcal{R}(g_i, g_i)^s$ for $1 \leqslant s \leqslant m_i - 1$. Then

$$\begin{aligned}1 &= \mathcal{R}(g_i, g_i^{m_i}) = \mathcal{R}(g_i, g_i)\mathcal{R}(g_i, g_i^{m_i-1})\frac{\omega(g_i, g_i, g_i^{m_i-1})}{\omega(g_i, g_i, g_i^{m_i-1})\omega(g_i, g_i^{m_i-1}, g_i)} \\ &= \mathcal{R}(g_i, g_i)^{m_i}\zeta_{m_i}^{-a_i}, \\ 1 &= \mathcal{R}(g_i^{m_i}, g_i) = \mathcal{R}(g_i^{m_i-1}, g_i)\mathcal{R}(g_i, g_i)\frac{\omega(g_i^{m_i-1}, g_i, g_i)\omega(g_i, g_i^{m_i-1}, g_i)}{\omega(g_i^{m_i-1}, g_i, g_i)} \\ &= \mathcal{R}(g_i, g_i)^{m_i}\zeta_{m_i}^{a_i}.\end{aligned}$$

Thus $r_{ii}^{m_i} = \zeta_{m_i}^{a_i} = \zeta_{m_i}^{-a_i}$.

Assume $i < j$. Applying (3.2) iteratively, one has $\mathcal{R}(g_i^k, g_j) = \mathcal{R}(g_i, g_j)^k$ for $1 \leqslant k \leqslant m_i - 1$. Therefore,

$$\begin{aligned}1 &= \mathcal{R}(g_i^{m_i}, g_j) = \mathcal{R}(g_i^{m_i-1}, g_j)\mathcal{R}(g_i, g_j)\frac{\omega(g_i^{m_i-1}, g_i, g_j)\omega(g_j, g_i^{m_i-1}, g_i)}{\omega(g_i^{m_i-1}, g_j, g_i)} \\ &= \mathcal{R}(g_i, g_j)^{m_i}.\end{aligned}$$

This implies that $r_{ij}^{m_i} = 1$. Applying (3.2) iteratively, one has $\mathcal{R}(g_i, g_j^k) = \mathcal{R}(g_i, g_j)^k$ for $1 \leqslant k \leqslant m_j - 1$. Therefore,

$$\begin{aligned}1 &= \mathcal{R}(g_i, g_j^{m_j}) = \mathcal{R}(g_i, g_j)\mathcal{R}(g_i, g_j^{m_j-1})\frac{\omega(g_j, g_i, g_j^{m_j-1})}{\omega(g_i, g_j, g_j^{m_j-1})\omega(g_j, g_j^{m_j-1}, g_i)} \\ &= r_{ij}^{m_j}\zeta_{m_i}^{-a_{ij}}.\end{aligned}$$

This implies that $r_{ij}^{m_j} = \zeta_{m_i}^{a_{ij}}$. Since $m_i|m_j$, we have $\zeta_{m_i}^{a_{ij}} = 1$. Since $0 \leqslant a_{ij} < m_i$, we arrive at $a_{ij} = 0$.

Assume $i > j$. Applying (3.2) iteratively, one has $\mathcal{R}(g_i^k, g_j) = \mathcal{R}(g_i, g_j)^k$ for $1 \leqslant k \leqslant m_i - 1$. Therefore,

$$\begin{aligned} 1 = \mathcal{R}(g_i^{m_i}, g_j) &= \mathcal{R}(g_i^{m_i-1}, g_j)\mathcal{R}(g_i, g_j)\frac{\omega(g_i^{m_i-1}, g_i, g_j)\omega(g_j, g_i^{m_i-1}, g_i)}{\omega(g_i^{m_i-1}, g_j, g_i)} \\ &= \mathcal{R}(g_i, g_j)^{m_i}\zeta_{m_j}^{a_{ij}}. \end{aligned}$$

This implies that $r_{ij}^{m_i} = \zeta_{m_j}^{-a_{ij}} = 1$. Applying (3.2) iteratively, one has $\mathcal{R}(g_i, g_j^k) = \mathcal{R}(g_i, g_j)^k$ for $1 \leqslant k \leqslant m_j - 1$. Therefore,

$$1 = \mathcal{R}(g_i, g_j^{m_j}) = \mathcal{R}(g_i, g_j)\mathcal{R}(g_i, g_j^{m_j-1})\frac{\omega(g_j, g_i, g_j^{m_j-1})}{\omega(g_i, g_j, g_j^{m_j-1})\omega(g_j, g_j^{m_j-1}, g_i)} = r_{ij}^{m_j}.$$

This implies that $r_{ij}^{m_j} = 1$.

The necessity is proved.

'$\Leftarrow$'. Conversely, define a map $\mathcal{R} : G \times G \longrightarrow k^*$ by setting

$$\mathcal{R}(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}) := \prod_{s=1}^{n} r_{ss}^{i_s j_s}.$$

We verify that $\mathcal{R}$ is a quasi-bicharacter of G with respect to ω.

Let $x = g_1^{i_1} \cdots g_n^{i_n},\ y = g_1^{j_1} \cdots g_n^{j_n},\ z = g_1^{k_1} \cdots g_n^{k_n}$, then

$$\mathcal{R}(g_1^{i_1} \cdots g_n^{i_n} \cdot g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}) = \prod_{s=1}^{n} r_{ss}^{(i_s+j_s)' k_s},$$

where $(i_s + j_s)'$ denotes the remainder of division of $i_s + j_s$ by m_s. Consider $\mathcal{R}(x, z)\mathcal{R}(y, z)\omega(z, x, y)\omega(x, y, z)/\omega(x, z, y)$. By direct calculation, one has

$$\frac{\omega(z, x, y)\omega(x, y, z)}{\omega(x, z, y)} = \prod_{l=1}^{n} \zeta_{m_l}^{a_l k_l[(i_l+j_l)/m_l]}.$$

Therefore,

$$\begin{aligned} &\mathcal{R}(x, z)\mathcal{R}(y, z)\frac{\omega(z, x, y)\omega(x, y, z)}{\omega(x, z, y)} \\ &= \prod_{s=1}^{n} r_{ss}^{(i_s+j_s)k_s} \prod_{l=1}^{n} \zeta_{m_l}^{a_l k_l[(i_l+j_l)/m_l]} \\ &= \prod_{s=1}^{n} r_{ss}^{((i_s+j_s)'+[(i_s+j_s)/m_s]m_s)k_s} \prod_{l=1}^{n} \zeta_{m_l}^{a_l k_l[(i_l+j_l)/m_l]} \\ &= \prod_{s=1}^{n} r_{ss}^{(i_s+j_s)' k_s} \prod_{l=1}^{n} c_{ll}^{[(i_l+j_l)/m_l]m_l k_l} \prod_{l=1}^{n} \zeta_{m_l}^{a_l k_l[(i_l+j_l)/m_l]} \\ &= \prod_{s=1}^{n} r_{ss}^{(i_s+j_s)' k_s} \prod_{l=1}^{n} \zeta_{m_l}^{-a_l k_l[(i_l+j_l)/m_l]} \prod_{l=1}^{n} \zeta_{m_l}^{a_l k_l[(i_l+j_l)/m_l]}. \end{aligned}$$

This implies that

$$\mathcal{R}(x,z)\mathcal{R}(y,z)\frac{\omega(z,x,y)\omega(x,y,z)}{\omega(x,z,y)} = \prod_{s=1}^{n} r_{ss}^{(i_s+j_s)'k_s} = \mathcal{R}(xy,z).$$

The sufficiency is proved. □

4. The DW invariant of the n-torus

In this section, we give a formula of the DW invariant for an arbitrary n-torus T^n associated with finite abelian groups. In the special case of $n = 2$, we recover and improve some results obtained in [25]. This is due to the fact that as we have an explicit formula of 2-cocycles, we are able to derive dimension formulas for irreducible projective representations of finite abelian groups. This is of independent interest.

4.1. Definition of DW invariants

Just as its name implies, such an invariant of 3-manifolds was introduced by Dijkgraaf and Witten in [5]. Then it was generalized to arbitrary dimension in [10] by Freed.

Now we recall briefly the definition of DW invariants. The reader is referred to [5, 10, 25] for more details. Let G be a finite group and let $[\omega] \in \mathrm{H}^n(BG; \Bbbk^*)$. For a closed oriented n-manifold M, the DW invariant of M is defined as

$$Z^{[\omega]}(M) = \frac{1}{|G|}\sum_{\phi:\pi_1(M)\to G} \langle f_\phi^*[\omega], [M]\rangle,$$

where $f_\phi : M \to BG$ is a map inducing ϕ on the fundamental group which is determined by ϕ up to homotopy, $[M]$ is the fundamental class of M, and $\langle , \rangle$ is the pairing $\mathrm{H}^n(M; \Bbbk^*) \times \mathrm{H}_n(M; \mathbb{Z}) \to \Bbbk^*$.

4.2. The DW invariant of the n-torus

The DW invariants of the n-torus for n-cocycle, implies the dimension of Hilbert space on an $(n-1)$-torus, T^{n-1}, (namely the number of ground states). This is a very important data for detecting the so-called 'topologically ordered' quantum states in physics. Some special cases were computed in [4, 16, 26–28, 31].

In particular, the thesis of de Wild Propitius [4] considers the 3-dimensional DW invariants, it discusses not only a few examples on the 3-torus partition function but also the Lens space $L(p, q)$ result in Chapter 2.9.

The calculations of twisted DW invariants, in terms of the mapping class group MCG data on d-torus T^d, as the 'quantum' representation of $\mathrm{MCG}(T^d) = SL(d, \mathbb{Z})$ generated by the so-called modular S and T data, have been recently explored, say in $2 + 1$ $(= 3)$-dimensions [16] and in 3+1 (=4)-dimensions [26, 27]. Roughly speaking, the computations of these S and T data, is analogous to the computation of a partition function obtained from gluing two states (vectors) living in the Hilbert space (a vector space) defined on $D^2 \times (S^1)^{d-1}$ (or other topology, where the manifold with a T^d-boundary is associated with a state in the Hilbert space) –

the gluing along the spatial T^d torus is glued via the mapping of S and T data of $\mathrm{MCG}(T^d) = SL(d, \mathbb{Z})$.

Let $\mathbb{Z}_d$ denote the quotient ring $\mathbb{Z}/d\mathbb{Z}$ and $M_n(\mathbb{Z}_d)$ the ring of $d \times d$ matrices with entries in $\mathbb{Z}_d$. Fix a d-th primitive root ξ of 1 and define

$$N_n(d) := \frac{\sum_{A \in M_n(\mathbb{Z}_d)} \xi^{\det A}}{d^n}.$$

LEMMA 4.1. *The function $N_n(d)$ is integer-valued and is multiplicative on d, that is, if $d = d_1 d_2$ with $(d_1, d_2) = 1$, then $N_n(d) = N_n(d_1)N_n(d_2)$. Moreover, for a prime p,*

$$N_n(p^m) = \sum_{i=1}^{m} p^{m(n-2)} p^{(m-i)(n-2)(n-1)} N_{n-1}(p^i)(p^{ni} - p^{n(i-1)}) + p^{m(n-2)n}.$$

Proof. Take $A = (a_{ij}) \in M_n(\mathbb{Z}_d)$. Then $\det A = a_{11}A_{11} + \cdots + a_{1n}A_{1n}$ where A_{ij} is the algebraic cofactor of a_{ij} and thus

$$\begin{aligned}\sum_{A \in M_n(\mathbb{Z}_d)} \xi^{\det A} &= \sum_{a_{11}=0}^{d-1} \xi^{a_{11}A_{11}} \cdots \sum_{a_{1n}=0}^{d-1} \xi^{a_{1n}A_{1n}} \\ &= d^n \#\{B \in M_{(n-1)\times n}(\mathbb{Z}_d) | \text{all } (n-1)\text{-minors of } B \text{ are } 0\}.\end{aligned}$$

Hence $N_n(d) = \#\{B \in M_{(n-1)\times n}(\mathbb{Z}_d) | \text{all } (n-1)\text{-minors of } B \text{ are } 0\}$.

Assume $B = (b_{ij}) \in M_{(n-1)\times n}(\mathbb{Z}_d)$ is such a matrix all of whose $(n-1)$-minors are 0. Define $\mathrm{ord}(b_{11}, \ldots, b_{1n})$ to be the smallest integer r such that $d | rb_{11}, \ \ldots, \ d | rb_{1n}$. Clearly, $\mathrm{ord}(b_{11}, \ldots, b_{1n}) | d$. Now suppose $d = p^m$ where p is prime. If $\mathrm{ord}(b_{11}, \ldots, b_{1n}) = p^i$, then $p^i b_{11} = p^m \widetilde{b_{11}}, \ \ldots, \ p^i b_{1n} = p^m \widetilde{b_{1n}}$. For $i \geqslant 1$, if $p | \widetilde{b_{11}}, \ \ldots, \ p | \widetilde{b_{1n}}$, then $\mathrm{ord}(b_{11}, \ldots, b_{1n}) \leqslant p^{i-1}$, contradiction. So we may assume, without loss of generality, that $p \nmid \widetilde{b_{11}}$. In this case, the matrix

$$P := \begin{pmatrix} \overline{\widetilde{b_{11}}} & \overline{\widetilde{b_{12}}} & \ldots & \overline{\widetilde{b_{1n}}} \\ \overline{0} & \overline{1} & \ldots & \overline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{0} & \overline{0} & \ldots & \overline{1} \end{pmatrix}$$

is invertible in $M_n(\mathbb{Z}_{p^m})$. Thus obviously, $(\overline{b_{11}}, \ldots, \overline{b_{1n}})P^{-1} = (\overline{p^{m-i}}, \overline{0}, \ldots, \overline{0})$. Assume $(\overline{b_{i1}}, \ldots, \overline{b_{in}})P^{-1} = (\overline{b'_{i1}}, \ldots, \overline{b'_{in}})$ for $i = 2, \ldots, n-1$. Since all $(n-1)$-minors of B are 0, all $(n-1)$-minors of

$$\begin{pmatrix} \overline{p^{m-i}} & \overline{0} & \ldots & \overline{0} \\ \overline{b'_{21}} & \overline{b'_{22}} & \ldots & \overline{b'_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b'_{n-1,1}} & \overline{b'_{n-1,2}} & \ldots & \overline{b'_{n-1,n}} \end{pmatrix}$$

are 0. Hence all $(n-2)$-minors of

$$\begin{pmatrix} \overline{b'_{22}} & \cdots & \overline{b'_{2n}} \\ \vdots & \ddots & \vdots \\ \overline{b'_{n-1,2}} & \cdots & \overline{b'_{n-1,n}} \end{pmatrix}$$

are 0 modulo p^i. So we have

$$N_n(p^m) = \sum_{i=1}^{m} p^{m(n-2)} p^{(m-i)(n-2)(n-1)} N_{n-1}(p^i)(p^{ni} - p^{n(i-1)}) + p^{m(n-2)n}.$$

Denote $S_n(d) = \{B \in M_{(n-1)\times n}(\mathbb{Z}_d) | \text{all } (n-1)\text{-minors of } B \text{ are } 0\}$. Then $B \mapsto (B \mod d_1, B \mod d_2)$ defines a map from $S_n(d_1 d_2)$ to $S_n(d_1) \times S_n(d_2)$. If $(d_1, d_2) = 1$, then this map is clearly injective and surjective by the Chinese Remainder theorem. Hence $N_n(d_1 d_2) = N_n(d_1) N_n(d_2)$. So if $d = p_1^{m_1} \cdots p_r^{m_r}$ where $p_1, \ldots, p_r$ are distinct primes, then $N_n(d) = N_n(p_1^{m_1}) \cdots N_n(p_r^{m_r})$. □

THEOREM 4.2. *The DW invariant of the n-torus T^n for a finite abelian group G is*

$$Z^{[\omega]}(T^n) = \frac{1}{|G|} \sum_{f_1,\ldots,f_n \in G} \frac{\prod_{\sigma\in A_n} \omega(f_{\sigma(1)}, \ldots, f_{\sigma(n)})}{\prod_{\sigma\in S_n \setminus A_n} \omega(f_{\sigma(1)}, \ldots, f_{\sigma(n)})}. \tag{4.1}$$

Let $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_l}$ where $m_i | m_{i+1}$ for $1 \leqslant i \leqslant l-1$. If $l < n$, then $Z^{[\omega]}(T^n) = |G|^{n-1}$. If $l = n$, then $Z^{[\omega]}(T^n) = |G|^{n-1}/d^{n(n-1)} N_n(d)$ where $d = m_1/(m_1, a_{1\cdots n})$. If $l > n$, then

$$Z^{[\omega]}(T^n) = \frac{1}{|G|} \sum_{A} \prod_{1\leqslant r_1 < \cdots < r_n \leqslant l} \zeta_{m_{r_1}}^{a_{r_1\cdots r_n} \det A\begin{pmatrix} 1 & \cdots & n \\ r_1 & \cdots & r_n \end{pmatrix}}$$

where $A = (\alpha_{ij})_{n\times l}$ and $0 \leqslant \alpha_{ij} < m_j$ for $1 \leqslant i \leqslant n$.

Proof. The n-torus T^n is obtained by gluing parallel edges of an n-dimensional cube. The cube can be subdivided into $n!$ n-simplexes such that each n-simplex has n successive edges in common with the cube. If we label the remaining n edges of the cube after gluing by $f_1, \ldots, f_n \in G$, then each n-simplex is uniquely determined by a permutation $(f'_1, \ldots, f'_n)$ of $(f_1, \ldots, f_n)$. The fundamental class $[T^n]$ is represented by an n-chain $\sigma : \Delta^n \to T^n$ where σ is the sum of those n-simplexes with the sign of which is positive if the permutation is even and negative otherwise. By [**13**, p89] we may identify $\mathrm{H}^n(BG; \Bbbk^*)$ and $\mathrm{H}^n(G; \Bbbk^*)$. Then when ϕ runs over all group homomorphisms from $\pi_1(T^n)$ to G, we have

$$\omega((f_\phi)_*([T^n])) = \frac{\prod_{\sigma\in A_n} \omega(f_{\sigma(1)}, \ldots, f_{\sigma(n)})}{\prod_{\sigma\in S_n\setminus A_n} \omega(f_{\sigma(1)}, \ldots, f_{\sigma(n)})}$$

where $f_1, \ldots, f_n$ run over G. Hence (4.1) holds.

Now let $f_i = (\alpha_{i1}, \ldots, \alpha_{il})$ where $0 \leqslant \alpha_{ij} < m_j$ for $1 \leqslant i \leqslant n$. Recall that

$$\omega(f_1, \ldots, f_n) = \prod_{k=1}^{n} \prod_{\substack{1\leqslant r_1<\cdots<r_k\leqslant l \\ \lambda_1+\cdots+\lambda_k=n, \lambda_1 \text{ odd} \\ \lambda_i\geqslant 1 \text{ for } 1\leqslant i\leqslant k}} \zeta_{m_{r_1}}^{(-1)^{\sum_{1\leqslant i<j\leqslant k}\lambda_i\lambda_j} \eta^{\alpha}_{r_1,[a_1,b_1]}\cdots\eta^{\alpha}_{r_k,[a_k,b_k]} a_{r_1^{\lambda_1}\cdots r_k^{\lambda_k}}}$$

where $a_k = 1,\ b_k = \lambda_k,\ \ldots,\ a_1 = \lambda_2 + \cdots + \lambda_k + 1,\ b_1 = \lambda_1 + \cdots + \lambda_k = n.$

$$\begin{aligned} Z^{[\omega]}(T^n) &= \frac{1}{|G|} \sum_{f_1,\ldots,f_n\in G} \frac{\prod_{\sigma\in A_n} \omega(f_{\sigma(1)}, \ldots, f_{\sigma(n)})}{\prod_{\sigma\in S_n\setminus A_n} \omega(f_{\sigma(1)}, \ldots, f_{\sigma(n)})} \\ &= \frac{1}{|G|} \sum_{f_1,\ldots,f_n\in G} \prod_{k=1}^{n} \prod_{\substack{1\leqslant r_1<\cdots<r_k\leqslant l \\ \lambda_1+\cdots+\lambda_k=n, \lambda_1 \text{ odd} \\ \lambda_i\geqslant 1 \text{ for } 1\leqslant i\leqslant k}} \\ &\quad \times \frac{\prod_{\sigma\in A_n} \zeta_{m_{r_1}}^{(-1)^{\sum_{1\leqslant i<j\leqslant k}\lambda_i\lambda_j} \eta^{\alpha}_{r_1,[\sigma(a_1),\sigma(b_1)]}\cdots\eta^{\alpha}_{r_k,[\sigma(a_k),\sigma(b_k)]} a_{r_1^{\lambda_1}\cdots r_k^{\lambda_k}}}}{\prod_{\sigma\in S_n\setminus A_n} \zeta_{m_{r_1}}^{(-1)^{\sum_{1\leqslant i<j\leqslant k}\lambda_i\lambda_j} \eta^{\alpha}_{r_1,[\sigma(a_1),\sigma(b_1)]}\cdots\eta^{\alpha}_{r_k,[\sigma(a_k),\sigma(b_k)]} a_{r_1^{\lambda_1}\cdots r_k^{\lambda_k}}}} \end{aligned}$$

If $(\lambda_1, \ldots, \lambda_k)$ is a partition of n and $\lambda_i > 1$ for some $i \in \{1, \ldots, k\}$, then $b_i > a_i$ and $\eta^{\alpha}_{r_i,[a_i,b_i]}$ contains the term $[(\alpha_{b_i,r_i} + \alpha_{b_i-1,r_i})/m_{r_i}]$. Let τ be the transposition $(b_i, b_i - 1)$, then

$$\begin{aligned} &\zeta_{m_{r_1}}^{(-1)^{\sum_{1\leqslant i<j\leqslant k}\lambda_i\lambda_j} \eta^{\alpha}_{r_1,[a_1,b_1]}\cdots\eta^{\alpha}_{r_k,[a_k,b_k]} a_{r_1^{\lambda_1}\cdots r_k^{\lambda_k}}} \\ &\quad = \zeta_{m_{r_1}}^{(-1)^{\sum_{1\leqslant i<j\leqslant k}\lambda_i\lambda_j} \eta^{\alpha}_{r_1,[\tau(a_1),\tau(b_1)]}\cdots\eta^{\alpha}_{r_k,[\tau(a_k),\tau(b_k)]} a_{r_1^{\lambda_1}\cdots r_k^{\lambda_k}}}. \end{aligned}$$

Hence

$$\frac{\prod_{\sigma\in A_n} \zeta_{m_{r_1}}^{(-1)^{\sum_{1\leqslant i<j\leqslant k}\lambda_i\lambda_j} \eta^{\alpha}_{r_1,[\sigma(a_1),\sigma(b_1)]}\cdots\eta^{\alpha}_{r_k,[\sigma(a_k),\sigma(b_k)]} a_{r_1^{\lambda_1}\cdots r_k^{\lambda_k}}}}{\prod_{\sigma\in S_n\setminus A_n} \zeta_{m_{r_1}}^{(-1)^{\sum_{1\leqslant i<j\leqslant k}\lambda_i\lambda_j} \eta^{\alpha}_{r_1,[\sigma(a_1),\sigma(b_1)]}\cdots\eta^{\alpha}_{r_k,[\sigma(a_k),\sigma(b_k)]} a_{r_1^{\lambda_1}\cdots r_k^{\lambda_k}}}} = 1.$$

Therefore,

$$\begin{aligned} Z^{[\omega]}(T^n) &= \frac{1}{|G|} \sum_{f_1,\ldots,f_n\in G} \prod_{1\leqslant r_1<\cdots<r_n\leqslant l} \\ &\quad \times \frac{\prod_{\sigma\in A_n} \zeta_{m_{r_1}}^{(-1)^{n(n-1)/2}\alpha_{\sigma(n),r_1}\cdots\alpha_{\sigma(1),r_n} a_{r_1\cdots r_n}}}{\prod_{\sigma\in S_n\setminus A_n} \zeta_{m_{r_1}}^{(-1)^{n(n-1)/2}\alpha_{\sigma(n),r_1}\cdots\alpha_{\sigma(1),r_n} a_{r_1\cdots r_n}}}. \end{aligned}$$

Hence if $l < n$, then each summand of (4.1) is 1 and $Z^{[\omega]}(T^n) = |G|^{n-1}$. If $l = n$, then equation (4.1) becomes

$$\begin{aligned} Z^{[\omega]}(T^n) &= \frac{1}{|G|}\sum_{A}(\zeta_{m_1}^{a_{1\cdots n}})^{\sum_{\sigma\in S_n}(-1)^{\text{sign of }\sigma}(-1)^{n(n-1)/2}\alpha_{\sigma(n)1}\cdots\alpha_{\sigma(1)n}} \\ &= \frac{1}{|G|}\sum_{A}(\zeta_{m_1}^{a_{1\cdots n}})^{\sum_{\sigma\in S_n}(-1)^{\text{sign of }\sigma}\alpha_{\sigma(1)1}\cdots\alpha_{\sigma(n)n}} \\ &= \frac{1}{|G|}\sum_{A}(\zeta_{m_1}^{a_{1\cdots n}})^{\det A} \end{aligned}$$

where $A = (\alpha_{ij})_{n\times l}$ and $0 \leqslant \alpha_{ij} < m_j$ for $1 \leqslant i \leqslant n$. Denote $\xi = \zeta_{m_1}^{a_{1\cdots n}}$, then ξ is a d-th primitive root of 1 where $d = m_1/(m_1, a_{1\cdots n})$. In this situation,

$$\begin{aligned} Z^{[\omega]}(T^n) &= \frac{1}{|G|}\sum_{A}\xi^{\det A} \\ &= \frac{1}{|G|}\left(\frac{m_1}{d}\cdots\frac{m_n}{d}\right)^n \sum_{A\in M_n(\mathbb{Z}_d)}\xi^{\det A} \\ &= \frac{|G|^{n-1}}{d^{n^2}}\sum_{A\in M_n(\mathbb{Z}_d)}\xi^{\det A}. \end{aligned}$$

If $l > n$, the formula is similarly derived as the case $l = n$. Hence lemma 4.1 completes the proof. □

4.3. The DW invariant of T^2 and projective representations

In [**25**] Turaev observed the connection between DW invariants of surfaces and projective representations of finite groups. In the case of T^2, our theorem 4.2 recovers some partial results of Turaev. Moreover, with a help of our explicit formula of 2-cocycles, we are able to derive a formula for the dimension of an arbitrary projective representation of finite abelian groups. This is of independent interest on the one hand, and helps to improve some formulas in [**25**] on the other hand.

Now let $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_l}$ with $m_1|m_2|\cdots|m_l$ and let ω be a 2-cocycle on G. In the case $k = 2$, (2.12) becomes

$$\omega(g_1^{i_1}\cdots g_l^{i_l}, g_1^{j_1}\cdots g_l^{j_l}) = \prod_{1\leqslant r<s\leqslant l}\zeta_{m_r}^{-a_{rs}i_s j_r}.$$

Let G_0 be the set of all ω-regular elements in G, i.e.,

$$G_0 = \{x \in G | \omega(x, y) = \omega(y, x) \text{ for all } y \in G\}.$$

It is well known that the number of irreducible ω-representations of G is $|G_0|$ and all irreducible ω-representations of G share a common dimension $d = \sqrt{|G|/|G_0|}$, see [**11**, **22**].

In the following we derive the formula of $|G_0|$, hence of d as well, in terms of the data $(a_{rs})_{1\leqslant r<s\leqslant l}$ of the given 2-cocycle ω. Consider the following antisymmetric $l\times l$-matrix

$$B=\begin{pmatrix} 0 & b_{12} & \dots & b_{1l} \\ -b_{12} & 0 & \dots & b_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{1l} & -b_{2l} & \dots & 0 \end{pmatrix}$$

where $b_{ij}=m_l/m_i a_{ij}$. Assume that the invariant factors of B are $\lambda_1,\ \dots,\ \lambda_k$ with $\lambda_1|\lambda_2|\cdots|\lambda_k$.

Proposition 4.3. *Keep the above notations. Then we have*

$$|G_0|=\frac{|G|}{m_l/(m_l,\lambda_1)\cdots m_l/(m_l,\lambda_k)},\qquad d=\sqrt{\frac{m_l}{(m_l,\lambda_1)}\cdots\frac{m_l}{(m_l,\lambda_k)}}.$$

Proof. By direct computations, we have

$$\begin{aligned}|G_0| &= \#\{(i_1,\dots,i_l)|0\leqslant i_r<m_r \text{ for } 1\leqslant r\leqslant l, \prod_{1\leqslant r<s\leqslant l}\zeta_{m_r}^{-a_{rs}i_sj_r}=\prod_{1\leqslant r<s\leqslant l}\zeta_{m_r}^{-a_{rs}i_rj_s}\\ &\qquad \text{for any } (j_1,\dots,j_l) \text{ where } 0\leqslant j_r<m_r \quad \text{for } 1\leqslant r\leqslant l\}\\ &= \frac{1}{m_l/m_1\cdots m_l/m_{l-1}}\#\{(i_1,\dots,i_l)|0\leqslant i_r<m_l \text{ for } 1\leqslant r\leqslant l, \begin{pmatrix} i_1 & \dots & i_l\end{pmatrix}\\ &\qquad \times B\begin{pmatrix} j_1\\ \vdots \\ j_l\end{pmatrix}\equiv 0 \mod m_l\\ &\qquad \text{for any } (j_1,\dots,j_l) \text{ where } 0\leqslant j_r<m_l \text{ for } 1\leqslant r\leqslant l\}\\ &= \frac{1}{m_l/m_1\cdots m_l/m_{l-1}}\#\{(i_1,\dots,i_l)|0\leqslant i_r<m_l \text{ for } 1\leqslant r\leqslant l, \lambda_r i_r\\ &\qquad \equiv 0 \mod m_l \text{ for } 1\leqslant r\leqslant k\}\\ &= \frac{1}{m_l/m_1\cdots m_l/m_{l-1}}(m_l,\lambda_1)\cdots(m_l,\lambda_k)m_l^{l-k}\\ &= \frac{|G|}{m_l/(m_l,\lambda_1)\cdots m_l/(m_l,\lambda_k)}\end{aligned}$$

and

$$d=\sqrt{\frac{|G|}{|G_0|}}=\sqrt{\frac{m_l}{(m_l,\lambda_1)}\cdots\frac{m_l}{(m_l,\lambda_k)}}.\qquad\square$$

We recover a result of Turaev [25] in the following

Corollary 4.4. *Keep the previous assumptions and notations. We have*

$$Z^{[\omega]}(T^2) = |G_0| = \frac{|G|}{m_l/(m_l,\lambda_1)\cdots m_l/(m_l,\lambda_k)}.$$

Proof. If $l \geqslant 2$, then

$$Z^{[\omega]}(T^2) = \sum_A \prod_{1\leqslant r_1<r_2\leqslant l} \zeta_{m_{r_1}}^{a_{r_1r_2}\det A\begin{pmatrix}1 & 2\\ r_1 & r_2\end{pmatrix}}$$

$$= \frac{1}{(m_l/m_1)^2\cdots(m_l/m_{l-1})^2} \sum_{0\leqslant\alpha_{ij}<m_l} \zeta_{m_l}^{\begin{pmatrix}\alpha_{11} & \dots & \alpha_{1l}\end{pmatrix} B\begin{pmatrix}\alpha_{21}\\ \vdots\\ \alpha_{2l}\end{pmatrix}}$$

where $A = \begin{pmatrix}\alpha_{11} & \dots & \alpha_{1l}\\ \alpha_{21} & \dots & \alpha_{2l}\end{pmatrix}$ and $0 \leqslant \alpha_{ij} < m_j$.

By assumption, there exist two invertible integral matrices $P,\ Q \in GL_l(\mathbb{Z})$ such that

$$B = P\begin{pmatrix}\lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0\end{pmatrix}Q.$$

Note that the images of P and Q in $M_l(\mathbb{Z}_{m_l})$ are also invertible. Hence

$$\sum_{0\leqslant\alpha_{ij}<m_l} \zeta_{m_l}^{\begin{pmatrix}\alpha_{11} & \dots & \alpha_{1l}\end{pmatrix} B\begin{pmatrix}\alpha_{21}\\ \vdots\\ \alpha_{2l}\end{pmatrix}}$$

$$= \sum_{0\leqslant\alpha_{ij}<m_l} \zeta_{m_l}^{\begin{pmatrix}\alpha_{11} & \dots & \alpha_{1l}\end{pmatrix}\begin{pmatrix}\lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0\end{pmatrix}\begin{pmatrix}\alpha_{21}\\ \vdots\\ \alpha_{2l}\end{pmatrix}}$$

$$= m_l^{2(l-k)} \sum_{\alpha_{11}=0}^{m_l-1}\sum_{\alpha_{21}=0}^{m_l-1} \zeta_{m_l}^{\alpha_{11}\lambda_1\alpha_{21}} \cdots \sum_{\alpha_{1k}=0}^{m_l-1}\sum_{\alpha_{2k}=0}^{m_l-1} \zeta_{m_l}^{\alpha_{1r}\lambda_r\alpha_{2r}}$$

It is well known that if $\xi^m = 1$, then

$$\sum_{i=0}^{m-1} \xi^{id} = \begin{cases} m, & \text{if } \xi^d = 1; \\ 0, & \text{if } \xi^d \neq 1. \end{cases}$$

Thus we have

$$\begin{aligned} Z^{[\omega]}(T^2) &= \frac{1}{|G|} \frac{1}{(m_l/m_1)^2 \cdots (m_l/m_{l-1})^2} m_l^{2(l-k)} m_l^k (m_l, \lambda_1) \cdots (m_l, \lambda_k) \\ &= \frac{|G|}{m_l/(m_l, \lambda_1) \cdots m_l/(m_l, \lambda_k)}. \end{aligned}$$

If $l = 1$, $Z^{[\omega]}(T^2) = |G| = |G_0|$. The conclusion also holds. □

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